

STRONG ULTRA-REGULARITY PROPERTIES FOR POSITIVE ELEMENTS IN THE TWISTED CONVOLUTIONS

YUANYUAN CHEN

ABSTRACT. We show that positive elements with respect to the twisted convolutions, belonging to some ultra-test function space of certain order at origin, belong to the ultra-test function space of the same order everywhere. We apply the result to positive semi-definite Weyl operators.

0. INTRODUCTION

Several issues in operator theory can be studied by means of the twisted convolution. For example, composition and positivity questions can be carried over to related questions for the twisted convolution product by simple manipulations. We notice the simple structure of the twisted convolution, since it essentially consists of a convolution product, disturbed by a (symplectic) Fourier kernel. It is also common that boundedness and regularity conditions on operator kernels often correspond to convenient conditions on related elements in the twisted convolution. For example, operator kernels which belong to the Schwartz space \mathcal{S} , or the Gelfand-Shilov spaces \mathcal{S}_s or Σ_s of Roumieu and Beurling types, respectively, carry over to elements in the same class in the twisted convolution. (See Section 1 for notations.)

In [8] it is shown that various kinds of singularities for positive elements with respect to the twisted convolution are attained at the origin. Furthermore, it is proved that regularity at origin for such elements impose global regularity and boundedness for these elements and their Fourier transforms.

More precisely, if $a \in \mathcal{D}'$ is positive semi-definite with respect to the twisted convolution, then it is proved that the following is true:

- (1) $a \in \mathcal{S}'$ (cf. [8, Theorem 2.6]);
- (2) if $\text{WF}_*(a)$ is any wave-front set of a and $(0, Y) \notin \text{WF}_*(a)$, then $(X, Y) \notin \text{WF}_*(a)$ and $(X, Y) \notin \text{WF}_*(\mathcal{F}_\sigma a)$. Here \mathcal{F}_σ is the symplectic Fourier transform (cf. [8, Theorem 4.14] and [9, Theorem 4.1]);

Key words and phrases. ultra-distributions, twisted convolution, Hermite series expansions, Weyl quantization.

- (3) if a is continuous at origin, then a and its Fourier transform \widehat{a} are continuous everywhere and belong to L^2 (cf. [8, Theorem 3.13]);
- (4) if $a \in C^\infty$ near origin, then $a \in \mathcal{S}$ (cf. [8, Theorem 3.13]);
- (5) if $s \geq 0$, $a \in C^\infty$ near origin and

$$|\partial^\alpha a(0)| \lesssim h^{|\alpha|} \alpha!^s \quad (0.1)$$

for some $h > 0$ (for every $h > 0$), then $a \in \mathcal{S}_s$ ($a \in \Sigma_s$) (cf. [1, Theorem 4.1]).

We note that if (0.1) holds true with $s < 1/2$ in (5), then a is trivially equal to 0, since the Gelfand-Shilov spaces \mathcal{S}_s and Σ_s are trivial for such choices of s .

In this paper we investigate related questions in background of Pilipović spaces, \mathcal{S}_s and Σ_s of Roumieu and Beurling type respectively, a family of function spaces which agrees with corresponding Gelfand-Shilov spaces when these are non-trivial (cf. [6, 7]). We introduce the so-called twisted Pilipović spaces $\mathcal{S}_{\sigma,s}$ and $\Sigma_{\sigma,s}$ which are symplectic analogies of Pilipović spaces, and show that they are homeomorphic to \mathcal{S}_s and Σ_s , respectively. We also show that

$$\mathcal{S}_{\sigma,s} = \mathcal{S}_s = \Sigma_s$$

when the right-hand side is non-trivial, and similarly for corresponding spaces of Beurling types.

We consider norm conditions of powers of a second order partial differential operator H_σ and its conjugate. These operators are symplectic analogies to certain partial harmonic oscillators. We show that H_σ and \bar{H}_σ commute and can be used to characterize $\mathcal{S}_{\sigma,s}$ and $\Sigma_{\sigma,s}$ as

$$a \in \mathcal{S}_{\sigma,s} \ (a \in \Sigma_{\sigma,s}) \quad \Leftrightarrow \quad \|H_\sigma^N \bar{H}_\sigma^N a\|_{L^\infty} \lesssim h^N (N!)^{4s} \quad (0.2)$$

for some $h > 0$ (for every $h > 0$). In Section 3 we show that if a is positive semi-definite with respect to the twisted convolution, then the relaxed condition

$$|H_\sigma^N \bar{H}_\sigma^N a(0)| \lesssim h^N (N!)^{4s}$$

of the right-hand of (0.2) is enough to ensure that a should belong to $\mathcal{S}_{\sigma,s}$ or $\Sigma_{\sigma,s}$.

1. PRELIMINARIES

In the first part we recall definitions of twisted convolution, the Weyl quantization and positivity in operator theory, and discuss basic properties. The verifications are in general omitted since they can be found in e.g. [8]. Thereafter we recall the definitions of Gelfand-Shilov and Pilipović spaces and discuss some properties. Here we also consider related symplectic analogies of such spaces, defined in terms of Wigner

distributions of Hermite functions, considered by Wong in [12, 13]. Finally we recall some results in [1] on positivity with respect to the twisted convolution.

1.1. Operators and positivity. Let a and b belong to $\mathcal{S}(\mathbf{R}^{2d})$, the set of Schwartz functions on \mathbf{R}^{2d} . Then the *twisted convolution* of a and b is given by

$$(a *_{\sigma} b)(X) = (2/\pi)^{d/2} \int_{\mathbf{R}^{2d}} a(X - Y)b(Y)e^{2i\sigma(X, Y)} dY.$$

Here σ is the symplectic form on $\mathbf{R}^d \times \mathbf{R}^d \simeq \mathbf{R}^{2d}$, given by

$$\sigma(X, Y) \equiv \langle y, \xi \rangle - \langle x, \eta \rangle, \quad X = (x, \xi) \in \mathbf{R}^{2d}, \quad Y = (y, \eta) \in \mathbf{R}^{2d}.$$

The definition of $*_{\sigma}$ extends in different ways. For example, the map $(a, b) \mapsto a *_{\sigma} b$ from $C_0^{\infty}(\mathbf{R}^{2d}) \times C_0^{\infty}(\mathbf{R}^{2d})$ to $C_0^{\infty}(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $\mathcal{S}'(\mathbf{R}^{2d}) \times \mathcal{S}(\mathbf{R}^{2d})$ to $\mathcal{S}'(\mathbf{R}^{2d})$, and from $\mathcal{D}'(\mathbf{R}^{2d}) \times C_0^{\infty}(\mathbf{R}^{2d})$ to $\mathcal{D}'(\mathbf{R}^{2d})$.

There are strong links between the twisted convolution, and continuity and composition properties in operator theory. This also includes analogous questions in the theory of pseudo-differential operators.

In fact, by straight-forward computations it follows that

$$A(a *_{\sigma} b) = (Aa) \circ (Ab), \quad (1.1)$$

where A is the operator defined by the formula

$$(Aa)(x, y) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} a((y - x)/2, \xi) e^{-i\langle x + y, \xi \rangle} d\xi. \quad (1.2)$$

(Here and in what follows we identify operators with their kernels.) We note that

$$(Aa)(x, y) = (\mathcal{F}^{-1}(a((y - x)/2, \cdot)))(-(x + y)),$$

where \mathcal{F} is the Fourier transform on $\mathcal{S}'(\mathbf{R}^d)$ which takes the form

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when $f \in \mathcal{S}(\mathbf{R}^d)$. Alternatively we may reformulate this identity as

$$(Aa)(x, y) = (\mathcal{F}_2^{-1}a)((y - x)/2, -(x + y)),$$

where $\mathcal{F}_2\Phi$ is the partial Fourier transform of $\Phi(x, y)$ with respect to the y -variable. Evidently, the mappings \mathcal{F}_2 and the pullback which takes $\Phi(x, y)$ into

$$\Phi((y - x)/2, -(x + y))$$

are homeomorphisms on $\mathcal{S}(\mathbf{R}^{2d})$ and on $\mathcal{S}'(\mathbf{R}^{2d})$, and unitary on $L^2(\mathbf{R}^{2d})$. Hence similar facts hold true for A .

From these mapping properties it follows that if $a \in \mathcal{S}'(\mathbf{R}^{2d})$, then Aa is a linear and continuous operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$. Furthermore, by the kernel theorem of Schwartz it follows that any linear

and continuous operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ is given by Aa , for a uniquely determined $a \in \mathcal{S}'(\mathbf{R}^{2d})$.

At this stage we also note that (1.1) remains true, if more generally, $a \in \mathcal{S}'(\mathbf{R}^{2d})$ and $b \in \mathcal{S}(\mathbf{R}^{2d})$, which follows by straight-forward computations.

The operator A can also in convenient ways be formulated in the framework of the Weyl calculus of pseudo-differential operators. More precisely, the Weyl quantization $\text{Op}^w(a)$ of $a \in \mathcal{S}(\mathbf{R}^{2d})$ (the symbol) is the operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}(\mathbf{R}^d)$ given by

$$\text{Op}^w(a)f(x) = (2\pi)^{-d} \iint_{\mathbf{R}^{2d}} a((x+y)/2, \xi) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi.$$

The definition of $\text{Op}^w(a)$ extends in continuous and similar ways as for Aa to any $\mathcal{S}'(\mathbf{R}^{2d})$, and then $\text{Op}^w(a)$ is continuous from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$. This extension can also be performed by the relation

$$\text{Op}^w(a) = (2\pi)^{-d/2} A(\mathcal{F}_\sigma a)$$

which follows by straight-forward computations. Here \mathcal{F}_σ is the symplectic Fourier transform on $\mathcal{S}'(\mathbf{R}^{2d})$, which takes the form

$$(\mathcal{F}_\sigma a)(X) \equiv \pi^{-d} \int_{\mathbf{R}^{2d}} a(Y) e^{2i\sigma(X, Y)} dY$$

when $a \in \mathcal{S}(\mathbf{R}^{2d})$.

From these facts it follow that the Weyl product $\#$, defined by

$$\text{Op}^w(a \# b) = \text{Op}^w(a) \circ \text{Op}^w(b)$$

is given by

$$a \# b = (2\pi)^{d/2} a *_\sigma (\mathcal{F}_\sigma b)$$

which again links the twisted convolution to compositions in operator theory.

There are also strong links between positivity for the twisted convolution and positivity in operator theory. We recall that a continuous and linear operator T from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ (from $C_0^\infty(\mathbf{R}^d)$ to $\mathcal{D}'(\mathbf{R}^d)$) is called positive semi-definite, whenever $(Tf, f) \geq 0$ for every $f \in \mathcal{S}(\mathbf{R}^d)$ ($f \in C_0^\infty(\mathbf{R}^d)$), and then we write $T \geq 0$. Since $C_0^\infty(\mathbf{R}^d)$ is dense in $\mathcal{S}(\mathbf{R}^d)$, it follows that an operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ is positive semi-definite, if it is positive semi-definite as an operator from $C_0^\infty(\mathbf{R}^d)$ to $\mathcal{D}'(\mathbf{R}^d)$.

Positivity for the twisted convolution is defined in an analogous way. That is, an element $a \in \mathcal{S}'(\mathbf{R}^{2d})$ ($a \in \mathcal{D}'(\mathbf{R}^{2d})$) is positive semi-definite with respect to the twisted convolution, whenever $(a *_\sigma \varphi, \varphi) \geq 0$ for every $\varphi \in \mathcal{S}(\mathbf{R}^{2d})$ ($\varphi \in C_0^\infty(\mathbf{R}^{2d})$). As above it follows that $a \in \mathcal{S}'(\mathbf{R}^{2d})$ is positive semi-definite with respect to $*_\sigma$, if it is positive semi-definite as an element in $\mathcal{D}'(\mathbf{R}^{2d})$.

The following proposition explains the links between positivity in operator theory and positivity for the twisted convolution. Here $W_{f,g}$

is the Wigner distribution of $f \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}'(\mathbf{R}^d)$, given by $W_{f,g} \equiv A^{-1}(\check{f} \otimes \bar{g})$. If $f, g \in \mathcal{S}(\mathbf{R}^d)$, then $W_{f,g}$ takes the form

$$W_{f,g}(x, \xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x - y/2) \overline{g(x + y/2)} e^{i\langle y, \xi \rangle} dy.$$

Proposition 1.1. *Let $a \in \mathcal{S}'(\mathbf{R}^{2d})$. Then the following conditions are equivalent:*

- (1) *a is positive semi-definite with respect to the twisted convolution;*
- (2) *Aa is a positive semi-definite operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$;*
- (3) *$\text{Op}^w(\mathcal{F}_\sigma a)$ is a positive semi-definite operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$;*
- (4) *$(\mathcal{F}_\sigma a, W_{f,f}) \geq 0$ for every $f \in \mathcal{S}(\mathbf{R}^d)$.*

1.2. Gelfand-Shilov spaces. Let $h, s \in \mathbf{R}_+$ be fixed. Then $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|} (\alpha! \beta!)^s}$$

is finite. Here the supremum is taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

The set $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which increases with h and s , and is contained in $\mathcal{S}(\mathbf{R}^d)$. If $s > 1/2$, then $\mathcal{S}_{s,h}$ and $\cup_{h>0} \mathcal{S}_{1/2,h}$ are dense in \mathcal{S} . Hence, the dual $(\mathcal{S}_{s,h})'(\mathbf{R}^d)$ of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbf{R}^d)$.

The *Gelfand-Shilov spaces* $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ with respect to $h > 0$. Consequently

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d),$$

The space $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with semi norms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$. Moreover, $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s \geq 1/2$, and $\Sigma_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s > 1/2$.

If $\varepsilon > 0$ and $s > 0$, then

$$\Sigma_s(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbf{R}^d).$$

The *Gelfand-Shilov distribution spaces* $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$ are the projective and inductive limits respectively of $\mathcal{S}'_{s,h}(\mathbf{R}^d)$. Hence

$$\mathcal{S}'_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d).$$

By [3], \mathcal{S}'_s and Σ'_s are the duals of \mathcal{S}_s and Σ_s , respectively.

The Gelfand-Shilov spaces and their duals are invariant under translations, dilations, (partial) Fourier transformations and under several other important transformations. In fact, by straight-forward computations it follows that the properties and results in Subsection 1.1 hold

true with \mathcal{S}_s and \mathcal{S}'_s in place of \mathcal{S} and \mathcal{S}' , respectively, when $s \geq 1/2$, or with Σ_s and Σ'_s in place of \mathcal{S} and \mathcal{S}' , respectively, when $s > 1/2$.

1.3. The Pilipović spaces. We start to consider spaces which are obtained by suitable estimates of Gelfand-Shilov or Gevrey type when using powers of the harmonic oscillator $H = |x|^2 - \Delta$, $x \in \mathbf{R}^d$. In general we omit the arguments, since more thorough exposition is available in e. g. [11].

Let $s \geq 0$ and $h > 0$. Then $\mathcal{S}_{h,s}(\mathbf{R}^d)$ is the Banach space which consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{h,s}} \equiv \sup_{N \geq 0} \frac{\|H^N f\|_{L^\infty}}{h^N (N!)^{2s}} < \infty. \quad (1.3)$$

If h_α is the Hermite function

$$h_\alpha(x) = \pi^{-\frac{d}{4}} (-1)^{|\alpha|} (2^{|\alpha|} \alpha!)^{-\frac{1}{2}} e^{\frac{|x|^2}{2}} (\partial^\alpha e^{-|x|^2}) \quad (1.4)$$

on \mathbf{R}^d of order α , then $H h_\alpha = (2|\alpha| + d) h_\alpha$. This implies that $\mathcal{S}_{h,s}(\mathbf{R}^d)$ contains all Hermite functions when $s > 0$, and if $s = 0$ and $\alpha \in \mathbf{N}^d$ satisfies $2|\alpha| + d \leq h$, then $h_\alpha \in \mathcal{S}_{h,s}(\mathbf{R}^d)$.

We let

$$\Sigma_s(\mathbf{R}^d) \equiv \bigcap_{h>0} \mathcal{S}_{h,s}(\mathbf{R}^d) \quad \text{and} \quad \mathcal{S}_s(\mathbf{R}^d) \equiv \bigcup_{h>0} \mathcal{S}_{h,s}(\mathbf{R}^d),$$

and equip these spaces by projective and inductive limit topologies, respectively, of $\mathcal{S}_{h,s}(\mathbf{R}^d)$, $h > 0$. (Cf. [4, 6, 7, 11].)

The space $\Sigma_s(\mathbf{R}^d)$ ¹ is called the *Pilipović space (of Beurling type) of order $s \geq 0$ on \mathbf{R}^d* . Similarly, $\mathcal{S}_s(\mathbf{R}^d)$ is called the *Pilipović space (of Roumieu type) of order $s \geq 0$ on \mathbf{R}^d* . Evidently, $\Sigma_0(\mathbf{R}^d)$ is trivially equal to $\{0\}$, while

$$h_\alpha \in \mathcal{S}_s(\mathbf{R}^d), \quad \text{when } s \geq 0 \quad \text{and} \quad h_\alpha \in \Sigma_s(\mathbf{R}^d), \quad \text{when } s > 0.$$

The dual spaces of $\mathcal{S}_{h,s}(\mathbf{R}^d)$, $\Sigma_s(\mathbf{R}^d)$ and $\mathcal{S}_s(\mathbf{R}^d)$ are denoted by $\mathcal{S}'_{h,s}(\mathbf{R}^d)$, $\Sigma'_s(\mathbf{R}^d)$ and $\mathcal{S}'_s(\mathbf{R}^d)$, respectively. We have

$$\Sigma'_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}'_{h,s}(\mathbf{R}^d)$$

when $s > 0$ and

$$\mathcal{S}'_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}'_{h,s}(\mathbf{R}^d)$$

when $s \geq 0$, with inductive respective projective limit topologies of $\mathcal{S}'_{h,s}(\mathbf{R}^d)$, $h > 0$ (cf. [11]).

¹The boldface characters Σ_s , \mathcal{S}_s , etc. denote Pilipović spaces, and non-boldface characters Σ_s , $\mathcal{F}\mathcal{S}_s$, etc. denote analogous Gelfand-Shilov spaces.

Let $s > 0$ and $\varepsilon > 0$. Then

$$\begin{aligned} \mathcal{S}_0(\mathbf{R}^d) &\subseteq \Sigma_s(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbf{R}^d) \subseteq \mathcal{S}(\mathbf{R}^d) \\ &\subseteq \mathcal{S}'(\mathbf{R}^d) \subseteq \Sigma'_{s+\varepsilon}(\mathbf{R}^d) \subseteq \mathcal{S}'_s(\mathbf{R}^d) \subseteq \Sigma'_s(\mathbf{R}^d) \subseteq \mathcal{S}'_0(\mathbf{R}^d). \end{aligned} \quad (1.5)$$

Furthermore, in [11] it is proved that $\mathcal{S}_0(\mathbf{R}^d)$ consists of all finite linear combinations of Hermite functions, while $\mathcal{S}'_0(\mathbf{R}^d)$ consists of all formal series

$$f = \sum_{\alpha \in \mathbf{N}^d} c_\alpha h_\alpha, \quad c_\alpha = c_\alpha(f) = (f, h_\alpha)_{L^2}. \quad (1.6)$$

The next propositions show that Pilipović spaces can be characterized by Hermite coefficients c_α given by (1.6). The proofs can be found in [2, 11]. Here $H_1 U$ and $H_2 U$ are the partial harmonic oscillators given by

$$H_1 U(x, y) = (|x|^2 - \Delta_x)U(x, y), \quad H_2 U(x, y) = (|y|^2 - \Delta_y)U(x, y). \quad (1.7)$$

Proposition 1.2. *Let $s \geq 0$ ($s > 0$) and $f \in \mathcal{S}'_0(\mathbf{R}^d)$ be given by (1.6). Then the following conditions are equivalent:*

- (1) $f \in \mathcal{S}_s(\mathbf{R}^d)$ ($f \in \Sigma_s(\mathbf{R}^d)$);
- (2) $|c_\alpha(f)| \lesssim e^{-r|\alpha|^{\frac{1}{2s}}}$ for some $r > 0$ (for every $r > 0$).

Proposition 1.3. *Let $p, q \in (0, \infty]$, $p_0 \in [1, \infty]$, $s \geq 0$ ($s > 0$), $U \in \mathcal{S}'_0(\mathbf{R}^{2d})$, and H_1 and H_2 be given by (1.7). Then the following conditions are equivalent:*

- (1) $U \in \mathcal{S}_s(\mathbf{R}^{2d})$ ($U \in \Sigma_s(\mathbf{R}^{2d})$);
- (2) $\|H_1^{N_1} H_2^{N_2} U\|_{L^{p_0}} \lesssim h^{N_1+N_2} (N_1! N_2!)^{2s}$ for some $h > 0$ (for every $h > 0$);
- (3) $\|H_1^{N_1} H_2^{N_2} U\|_{M^{p,q}} \lesssim h^{N_1+N_2} (N_1! N_2!)^{2s}$ for some $h > 0$ (for every $h > 0$).

Remark 1.4. Let \mathcal{S}_s and Σ_s be the Gelfand-Shilov spaces of order $s \geq 0$. Then it is proved in [6, 7] that

$$\mathcal{S}_{s_1} = \mathcal{S}_{s_1}, \quad \Sigma_{s_2} = \Sigma_{s_2}, \quad s_1 \geq \frac{1}{2}, \quad s_2 > \frac{1}{2}$$

and

$$\mathcal{S}_{s_1} \neq \mathcal{S}_{s_1} = \{0\}, \quad \Sigma_{s_2} \neq \Sigma_{s_2} = \{0\}, \quad s_1 < \frac{1}{2}, \quad 0 < s_2 \leq \frac{1}{2}.$$

Remark 1.5. In [11] it is proved that \mathcal{S}_{s_1} and Σ_{s_2} are not invariant under dilations when $s_1 < 1/2$ and $s_2 \leq 1/2$.

Remark 1.6. Let the hypothesis in Proposition 1.3 be fulfilled. By letting $N_1 = N_2 = N$ we get

$$(2)' \quad \|H_1^N H_2^N U\|_{L^{p_0}} \lesssim h^N N!^{4s} \text{ for some } h > 0 \text{ (for every } h > 0);$$

(3)' $\|H_1^N H_2^N U\|_{M^{p,q}} \lesssim h^N N!^{4s}$ for some $h > 0$ (for every $h > 0$).

The same arguments as in [2, 11] imply that these conditions are equivalent. Furthermore, let $\tilde{\mathcal{S}}_s(\mathbf{R}^{2d})$ ($\tilde{\Sigma}_s(\mathbf{R}^{2d})$) be the set of all $U \in \mathcal{S}'_0(\mathbf{R}^{2d})$ such that

$$|c_\alpha(U)| \lesssim e^{-r(\langle \alpha_1 \rangle \langle \alpha_2 \rangle)^{\frac{1}{4s}}}, \quad \alpha = (\alpha_1, \alpha_2),$$

for some $r > 0$ (for every $r > 0$). Then it follows by similar arguments as in [2, 11] that

$$(2)' \Leftrightarrow (3)' \Leftrightarrow U \in \tilde{\mathcal{S}}_s(\mathbf{R}^{2d}) \quad (U \in \tilde{\Sigma}_s(\mathbf{R}^{2d})).$$

We note that $\mathcal{S}_s \subseteq \tilde{\mathcal{S}}_s \subseteq \mathcal{S}_{2s}$ with strict inclusions.

2. TWISTED PILIPOVIĆ SPACES AND THEIR PROPERTIES

In this section we introduce twisted Pilipović spaces as the counter images of the operator A on Pilipović spaces, and deduce some basic properties. We also consider their distribution spaces.

We begin with some definitions.

Definition 2.1. The *Hermite-Wong function* of order

$$\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^d \times \mathbf{N}^d \simeq \mathbf{N}^{2d}$$

on \mathbf{R}^{2d} is given by

$$\varrho_\alpha \equiv A^{-1}(h_{\alpha_1} \otimes h_{\alpha_2}) = A^{-1}(h_{\alpha_1} \otimes \overline{h_{\alpha_2}}) = (-1)^{|\alpha_1|} W_{h_{\alpha_1}, h_{\alpha_2}}.$$

The Hermite-Wong functions were studied in different ways by M. W. Wong in [12, 13]. By the definition it follows that

$$\varrho_\alpha(X) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} h_{\alpha_1}\left(\frac{y}{2} - x\right) \overline{h_{\alpha_2}\left(\frac{y}{2} + x\right)} e^{i\langle y, \xi \rangle} dy,$$

when $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^{2d}$ and $X = (x, \xi) \in \mathbf{R}^{2d}$.

We observe that the Hermite-Wong functions are eigenfunctions to \mathcal{F}_σ . More precisely, we have

$$\mathcal{F}_\sigma \varrho_{\alpha_1, \alpha_2} = (-1)^{|\alpha_1|} \varrho_{\alpha_1, \alpha_2},$$

which follows from the fact that $\mathcal{F}_\sigma(W_{f,g}) = W_{\check{f},g}$ (see e. g. [?, 5]. Here $\check{f}(x) = f(-x)$).

Definition 2.2. Let $s > 0$.

(1) The set $\mathcal{S}'_{\sigma,0}(\mathbf{R}^{2d})$ consists of all formal expansions

$$a = \sum_{\alpha} c_{\alpha} \varrho_{\alpha}, \tag{2.1}$$

where $\{c_{\alpha}\}_{\alpha \in \mathbf{N}^{2d}} \subseteq \mathbf{C}$.

(2) The set $\mathcal{S}_{\sigma,0}(\mathbf{R}^{2d})$ consists of all expansions in (2.1) such that c_{α} are non-zero for at most finite numbers of α .

- (3) The set $\mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($\Sigma_{\sigma,s}(\mathbf{R}^{2d})$) consists of all expansions in (2.1) such that

$$|c_\alpha| \lesssim e^{-c|\alpha|^{\frac{1}{2s}}}$$

for some $c > 0$ (for every $c > 0$).

- (4) The set $\mathcal{S}'_{\sigma,s}(\mathbf{R}^{2d})$ ($\Sigma'_{\sigma,s}(\mathbf{R}^{2d})$) consists of all expansions in (2.1) such that

$$|c_\alpha| \lesssim e^{c|\alpha|^{\frac{1}{2s}}}$$

for every $c > 0$ (for some $c > 0$).

The spaces in Definition 2.2 are equipped by topologies in similar way as for the Pilipovič spaces in [11].

The set $\mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($\Sigma_{\sigma,s}(\mathbf{R}^{2d})$) is called the *twisted Pilipovič space of Roumieu type (Beurling type)* of order s . It follows that the sets $\mathcal{S}'_{\sigma,s}(\mathbf{R}^{2d})$ and $\Sigma'_{\sigma,s}(\mathbf{R}^{2d})$ are corresponding distribution spaces, since similar facts hold true for Pilipovič space [11].

We extend the definition of A on \mathcal{S} by letting

$$Aa = \sum_{\alpha} c_{\alpha} h_{\alpha}$$

when $a \in \mathcal{S}'_{\sigma,0}(\mathbf{R}^{2d})$ is giving by (2.1). It follows that A is a homeomorphism from $\mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_s(\mathbf{R}^{2d})$, from $\Sigma_{\sigma,s}(\mathbf{R}^{2d})$ to $\Sigma_s(\mathbf{R}^{2d})$, and similarly for their duals. Since it is clear that A is homeomorphism on any Fourier invariant Gelfand-Shilov spaces, we get

$$\mathcal{S}_{\sigma,s}(\mathbf{R}^{2d}) = \mathcal{S}_s(\mathbf{R}^{2d}) = \mathcal{S}_s(\mathbf{R}^{2d}), \quad \text{when } s \geq 1/2$$

and

$$\Sigma_{\sigma,s}(\mathbf{R}^{2d}) = \Sigma_s(\mathbf{R}^{2d}) = \Sigma_s(\mathbf{R}^{2d}), \quad \text{when } s > 1/2,$$

and similarly for corresponding distribution spaces.

Remark 2.3. Let $a \in \mathcal{S}'_{\sigma,0}(\mathbf{R}^{2d})$ be as in (2.1). Since A is a homeomorphism on $\mathcal{S}(\mathbf{R}^{2d})$ and on $\mathcal{S}'(\mathbf{R}^{2d})$, it follows from [10] that a belongs to $\mathcal{S}(\mathbf{R}^{2d})$ if and only if $c_{\alpha} \lesssim \langle x \rangle^{-N}$ for every $N \geq 0$. In the same way, $a \in \mathcal{S}'(\mathbf{R}^{2d})$ if and only if $c_{\alpha} \lesssim \langle x \rangle^N$ for some $N \geq 0$.

Next we discuss the partial harmonic oscillators H_1 and H_2 in Proposition 1.3, and their counter images under the operator A . We let H_{σ} be the operator on $\mathcal{S}(\mathbf{R}^{2d})$, given by

$$H_{\sigma} = (|X|^2 - \frac{1}{4}\Delta_X) + \langle \xi, D_x \rangle - \langle x, D_{\xi} \rangle, \quad X = (x, \xi) \in \mathbf{R}^{2d},$$

and we let $T_{\sigma} = H_{\sigma} \circ \bar{H}_{\sigma}$. Here we note that

$$\bar{H}_{\sigma} = (|X|^2 - \frac{1}{4}\Delta_X) - \langle \xi, D_x \rangle + \langle x, D_{\xi} \rangle.$$

The following lemma explains some spectral properties of the considered operators.

Lemma 2.4. *Let $s \geq 0$. Then the following is true:*

- (1) *the Hermite-Wong functions ϱ_α are eigenfunctions to H_σ , \bar{H}_σ and T_σ , and*

$$H_\sigma \varrho_{\alpha_1, \alpha_2} = (2|\alpha_1| + d) \varrho_{\alpha_1, \alpha_2}, \quad \bar{H}_\sigma \varrho_{\alpha_1, \alpha_2} = (2|\alpha_2| + d) \varrho_{\alpha_1, \alpha_2}, \quad (2.2)$$

and

$$T_\sigma \varrho_{\alpha_1, \alpha_2} = (2|\alpha_1| + d)(2|\alpha_2| + d) \varrho_{\alpha_1, \alpha_2};$$

- (2) *H_σ and \bar{H}_σ restrict to homeomorphisms on $\mathcal{S}_{\sigma, s}(\mathbf{R}^{2d})$ and on $\Sigma_{\sigma, s}(\mathbf{R}^{2d})$;*
(3) *the definitions of H_σ and \bar{H}_σ extend uniquely to homeomorphisms on $\mathcal{S}'(\mathbf{R}^{2d})$, $\mathcal{S}'_{\sigma, s}(\mathbf{R}^{2d})$ and on $\Sigma'_{\sigma, s}(\mathbf{R}^{2d})$.*

For the proof, we shall make use of the operators

$$\begin{aligned} Z_{1,j} &= \frac{1}{2} \partial_{z_j} + \bar{z}_j, & \tilde{Z}_{1,j} &= \frac{1}{2} \partial_{\bar{z}_j} - z_j, \\ Z_{2,j} &= \frac{1}{2} \partial_{\bar{z}_j} + z_j, & \tilde{Z}_{2,j} &= \frac{1}{2} \partial_{z_j} - \bar{z}_j, \end{aligned}$$

where

$$z_j = x_j + i\xi_j, \quad \bar{z}_j = x_j - i\xi_j,$$

$$\partial_{z_j} = \partial_{x_j} - i\partial_{\xi_j}, \quad \partial_{\bar{z}_j} = \partial_{x_j} + i\partial_{\xi_j},$$

(see [13, Section 22]). By similar arguments as in the proof of Theorem 22.1 in [13] we get

$$\begin{aligned} Z_{1,j} \varrho_{\alpha_1, \alpha_2} &= (2|\alpha_{2,j}|)^{1/2} \varrho_{\alpha_1, \alpha_2 - e_j}, \\ \tilde{Z}_{1,j} \varrho_{\alpha_1, \alpha_2} &= -(2|\alpha_{2,j}| + 2)^{1/2} \varrho_{\alpha_1, \alpha_2 + e_j}, \\ Z_{2,j} \varrho_{\alpha_1, \alpha_2} &= -(2|\alpha_{1,j}|)^{1/2} \varrho_{\alpha_1 - e_j, \alpha_2}, \\ \tilde{Z}_{2,j} \varrho_{\alpha_1, \alpha_2} &= (2|\alpha_{1,j}| + 2)^{1/2} \varrho_{\alpha_1 + e_j, \alpha_2}, \end{aligned} \quad (2.3)$$

where e_1, \dots, e_d is the standard basis in \mathbf{R}^d , i.e. $e_j = (\delta_{1,j}, \dots, \delta_{d,j})$, $j = 1, \dots, d$, and $\delta_{i,j}$ is the Kroniker's delta function.

In view of (2.3), the operators $Z_{1,j}$ and $Z_{2,j}$ can be considered as symplectic analogies of annihilation operators, $\tilde{Z}_{1,j}$ and $\tilde{Z}_{2,j}$ as symplectic analogies of creation operators.

Proof. First we prove (1). By straight-forward computations, we obtain

$$H_\sigma = -\frac{1}{2} \left(\sum_j Z_{2,j} \tilde{Z}_{2,j} + \tilde{Z}_{2,j} Z_{2,j} \right)$$

and

$$\bar{H}_\sigma = -\frac{1}{2} \left(\sum_j Z_{1,j} \tilde{Z}_{1,j} + \tilde{Z}_{1,j} Z_{1,j} \right).$$

Hence, by (2.3) we get

$$H_\sigma \varrho_{\alpha_1, \alpha_2} = (2|\alpha_1| + d) \varrho_{\alpha_1, \alpha_2},$$

and

$$\bar{H}_\sigma \varrho_{\alpha_1, \alpha_2} = (2|\alpha_2| + d) \varrho_{\alpha_1, \alpha_2},$$

and (1) follows.

By (2.2), it follows that H_σ and \bar{H}_σ restrict to homeomorphisms on $\mathcal{S}_{\sigma, s}(\mathbf{R}^{2d})$ and on $\Sigma_{\sigma, s}(\mathbf{R}^{2d})$, which gives (2).

If $a \in \mathcal{S}'_{\sigma, s}(\mathbf{R}^{2d})$ and $b \in \mathcal{S}_{\sigma, s}(\mathbf{R}^{2d})$. We now let H_σ be defined by

$$(H_\sigma a, b)_{L^2} = (a, \bar{H}_\sigma b)_{L^2},$$

as usual, which extends the definitions of H_σ and \bar{H}_σ to $\mathcal{S}'_{\sigma, s}(\mathbf{R}^{2d})$. The extensions of these operators to $\Sigma'_{\sigma, s}(\mathbf{R}^{2d})$ and $\mathcal{S}'(\mathbf{R}^{2d})$ are performed in similar ways. By (2.2), it follows that these extensions are unique. \square

The next lemma shows important links between the latter operators and partial harmonic oscillators.

Lemma 2.5. *Let H_1 and H_2 be as in Proposition 1.3, and let $a \in \mathcal{S}_{\sigma, s}(\mathbf{R}^{2d})$. Then H_σ and \bar{H}_σ are commuting to each other, and*

$$A(H_\sigma^{N_1} \bar{H}_\sigma^{N_2} a) = H_1^{N_1} H_2^{N_2} (Aa), \quad (2.4)$$

for every interger $N_1, N_2 \geq 0$. In particular, if $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are sequences in $l^2(\mathbf{N}; L^2(\mathbf{R}^d))$, and a is given by

$$a = \sum_{k=0}^\infty A^{-1}(f_k \otimes \bar{g}_k),$$

then

$$A(T_\sigma^N a) = \sum_{k=0}^\infty (H^N f_k) \otimes (\overline{H^N g_k}),$$

where the series convergences in $\mathcal{S}'(\mathbf{R}^{2d})$.

Proof. The commutation between H_σ and \bar{H}_σ follows if we prove (2.4). We recall the operators

$$\begin{aligned} P_j &= \frac{1}{2i} \partial_{\xi_j} - x_j, & \Pi_j &= \frac{1}{2i} \partial_{x_j} + \xi_j, \\ T_j &= \frac{1}{2i} \partial_{\xi_j} + x_j, & \Theta_j &= \frac{1}{2i} \partial_{x_j} - \xi_j, \end{aligned} \quad (2.5)$$

and the relations

$$\begin{aligned} A(P_j^2 a) &= x_j^2 Aa, & A(\Pi_j^2 a) &= -\partial_{x_j}^2 (Aa), \\ A(T_j^2 a) &= y_j^2 Aa, & A(\Theta_j^2 a) &= -\partial_{y_j}^2 (Aa), \end{aligned} \quad (2.6)$$

from [1, Theorem 4.1].

By straight-forward computations we get

$$(x_j^2 - \partial_{x_j}^2)(Aa) = A((P_j^2 + \Pi_j^2)a) = A(H_{\sigma,j}a),$$

where $H_{\sigma,j} = (X_j^2 - \frac{1}{4}\Delta_{X_j}) + \xi_j D_{x_j} - x_j D_{\xi_j}$.

Summing up over all j gives

$$H_1(Aa) = A(H_\sigma a).$$

In the same way we get

$$H_2(Aa) = A(\bar{H}_\sigma a),$$

and the result follows by induction. \square

From these mapping properties, Proposition 1.3 can now be carried over to the case of twisted Pilipović spaces as follows.

Proposition 2.6. *Let $p, q \in (0, \infty]$ and $p_0 \in [1, \infty]$ and let $s \geq 0$ ($s > 0$). Then the following conditions are equivalent.*

- (1) $a \in \mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbf{R}^{2d})$);
- (2) $\|H_\sigma^{N_1} \bar{H}_\sigma^{N_2} a\|_{L^{p_0}} \lesssim h^{N_1+N_2} (N_1! N_2!)^{2s}$ for some $h > 0$ (for every $h > 0$);
- (3) $\|H_\sigma^{N_1} \bar{H}_\sigma^{N_2} a\|_{M^{p,q}} \lesssim h^{N_1+N_2} (N_1! N_2!)^{2s}$ for some $h > 0$ (for every $h > 0$).

Proof. Let $U = Aa$. Since $M^{p_1}(\mathbf{R}^{2d}) \subseteq M^{p,q}(\mathbf{R}^{2d}) \subseteq M^{p_2}(\mathbf{R}^{2d})$, when $p_1 = \min(p, q)$ and $p_2 = \max(p, q)$, we may assume that $p = q$.

Since A is a homeomorphism on $M^p(\mathbf{R}^{2d})$, we get

$$\|H_\sigma^{N_1} \bar{H}_\sigma^{N_2} a\|_{M^p} = \|A(H_\sigma^{N_1} \bar{H}_\sigma^{N_2} a)\|_{M^p} = \|H_1^{N_1} H_2^{N_2} U\|_{M^p},$$

and the equivalence between (3) and Proposition 1.3 (3) follows. The equivalence between (1) and (3) now follows from Proposition 1.3 and the fact that A is a homeomorphism from $\mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ to $\mathcal{S}_s(\mathbf{R}^{2d})$.

Finally by the embeddings

$$M^1(\mathbf{R}^{2d}) \subseteq L^{p_0}(\mathbf{R}^{2d}) \subseteq M^\infty(\mathbf{R}^{2d}),$$

the equivalence between (2) and (3) now follows. \square

Corollary 2.7. *If $s \geq 0$ and $a \in \mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbf{R}^{2d})$), then*

$$\|T_\sigma^N a\|_{L^\infty} \lesssim h^{2N} (N!)^{4s}, \quad (2.7)$$

holds for some $h > 0$ (for every $h > 0$).

Remark 1.6 and Lemma 2.5 show that (2.7) is necessary but not sufficient in order for $a \in \mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ or $a \in \Sigma_{\sigma,s}(\mathbf{R}^{2d})$.

3. TWISTED PILIPOVIĆ SPACE PROPERTY FOR POSITIVE ELEMENTS WITH RESPECT TO THE TWISTED CONVOLUTION

We study positive elements with respect to twisted convolution in \mathcal{S}' , having the twisted Pilipović space regularities near the origin. We show that such elements are in $\mathcal{S}_{\sigma,s}$ or in $\Sigma_{\sigma,s}$.

The following theorem shows that the condition of the form (2.7) at origin is sufficient that the converse of Corollary 2.7 holds when dealing with positive semi-definite elements with respect to the twisted convolution.

Theorem 3.1. *Let $s \geq 0$, $a \in \mathcal{S}'(\mathbf{R}^{2d})$ and $(a *_\sigma \psi, \psi) \geq 0$ for every $\psi \in \mathcal{S}(\mathbf{R}^{2d})$. If*

$$(T_\sigma^N a)(0, 0) \lesssim h^{2N} (N!)^{4s},$$

holds for some $h > 0$ (for every $h > 0$), then $a \in \mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbf{R}^{2d})$).

Proof. By the assumption, we may write $a = \sum_k A^{-1}(f_k \otimes \overline{f_k})$. By Lemma 2.5, we obtain

$$A(T_\sigma^N a) = \sum_k (H^N f_k \otimes \overline{H^N f_k}),$$

for some sequence $\{f_k\}_{k=0}^\infty$.

Let $K = \sum_k f_k \otimes \overline{f_k}$ be the kernel of Aa . Then

$$\begin{aligned} \|H_1^N H_2^N K\|_{L^2} &\leq \|H_1^N H_2^N K\|_{\text{Tr}} \\ &= \|A(T_\sigma^N a)\|_{\text{Tr}} = \sum_k \|H^N f_k\|_{L^2}^2 = (\pi/2)^{d/2} (T_\sigma^N a)(0, 0). \end{aligned}$$

Thus by the assumption, we get

$$\|H_1^N H_2^N K\|_{L^2} \lesssim h^{2N} (N!)^{4s},$$

for some $h > 0$ (for every $h > 0$), giving that $K \in \mathcal{S}_s(\mathbf{R}^{2d})$ ($K \in \Sigma_s(\mathbf{R}^{2d})$) in view of Proposition 1.3 and Remark 1.6. Hence $a \in \mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbf{R}^{2d})$). \square

Proposition 3.2. *Let $s \geq 0$ be real, and let $a \in \mathcal{S}'(\mathbf{R}^{2d})$ be such that $\text{Op}^\omega(a) \geq 0$. If*

$$(T_\sigma^N(\mathcal{F}_\sigma a))(0) \lesssim h^{2N} (N!)^{4s}, \quad (3.1)$$

holds for some $h > 0$ (for every $h > 0$), then $a \in \mathcal{S}_{\sigma,s}(\mathbf{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbf{R}^{2d})$).

REFERENCES

- [1] Y. Chen, J. Toft, *Boundeness of Gevrey and Gelfand-Shilov kernels of positive semi-definite operators*. J. Pseudo-Differ. Oper. Appl. 6 (2015): 153-185.
- [2] Y. Chen, M. Signahl, J. Toft, *Factorizations and singular value estimates of operators with Gelfand-Shilov and Piliopović kernels*, arXiv:1511.06257.

- [3] I. M. Gel'fand, G. E. Shilov, *Generalized functions*, Vol.2, Academic press, Boston, 1968.
- [4] T. Gramchev, S. Pilipović, L. Rodino *Classes of degenerate elliptic operators in Gelfand-Shilov spaces in: L. Rodino, M. W. Wong (Eds) New developments in pseudo-differential operators*, Operator Theory: Advances and Applications **189**, Birkhäuser Verlag, Basel 2009, pp. 15-31.
- [5] G. B. Folland, *Harmonic Analysis in Phase Space.*, Princeton University Press, Princeton (1989).
- [6] S. Pilipović *Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions*, SIAM J. Math. Anal. **17** (1986), 477-484.
- [7] S. Pilipović *Tempered ultradistributions*, Boll. U.M.I. **7** (1988), 235-251.
- [8] J. Toft, *Positivity properties in noncommutative convolution algebras with applications in pseudo-differential calculus*, Bull. Sci. math. **127** (2003):101-132.
- [9] J. Toft, *Wave front set of positive operators and for positive elements in non-commutative convolution algebras*, Studia Math. **179** (2007), 63-80.
- [10] J. Toft, *Multiplication properties in Gelfand-Shilov pseudo-differential calculus* in: S. Molahajlo, S. Pilipović, J. Toft, M. W. Wong (Eds), *Pseudo- Differential Operators, Generalized Functions and Asymptotics, Operator Theory: Advances and Applications 231*, Birkhäuser, Basel Heidelberg NewYork Dordrecht London, 2013, pp. 117-172.
- [11] J. Toft *Images of function and distribution spaces under the Bargmann transform*, arXiv:1409.5238 (preprint).
- [12] M. W. Wong, *Weyl transforms and a degenerate elliptic partial differential equation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), 3863-3870.
- [13] M. W. Wong, *Weyl transforms*, Springer-Verlag New York.

DEPARTMENT OF COMPUTER SCIENCE, PHYSICS AND MATHEMATICS, LINNÆUS UNIVERSITY, SWEDEN

E-mail address: `yuanyuan.chen@lnu.se`